# HÀM SỐLÔGARIT

**1)** $\log_a 1 = 0$ **2)** $\log_a a = 1$ **3)** $\log_a b + \log_a c = \log_a (bc)$ **4)** $\log_a b - \log_a c = \log_a \dfrac{b}{c}$

**5)** $a^{\log_a b} = b$ **6)** $\log_{a^\alpha} b^\beta = \dfrac{\beta}{\alpha} \log_a b \rightarrow \begin{cases} \log_a b^\beta = \beta \log_a b \\ \log_{a^\alpha} b = \dfrac{1}{\alpha} \log_a b \end{cases}$

**7)** $\log_a b.\log_b c = \log_a c \rightarrow \begin{cases} \log_a b.\log_b a = 1 \rightarrow \log_a b = \dfrac{1}{\log_b a} \\ \log_b c = \dfrac{\log_a c}{\log_a b} \end{cases}$

**Chú ý:** +) **Lôgarit thập phân** : $\log_{10} b = \log b = \lg b$

+) **Lôgarit tự nhiên ( lôgarit Nêpe) :** $\log_e b \quad \ln b$ **(** $e \approx 2,71828$ **)**

## A. CÁC VÍ DỤ MINH HỌA

**Ví dụ 1**: Tính giá trị các biểu thức sau:

**1) A =** $\log_3\left(\log_{2\sqrt{2}} \sqrt[3]{\sqrt{2}}\right)$ **2) B =** $\log_{\sqrt{6}} 3.\log_3 36$ **3) C =** $\log_{\frac{1}{3}} 5.\log_{25} \dfrac{1}{27}$

**4) D =** $\left(\sqrt[3]{9}\right)^{\frac{3}{2\log_5 3}}$ **5) E=** $25^{\frac{1}{2}+\frac{1}{9}\log_{\frac{1}{5}} 27+\log_{125} 81}$ **6) F =** $\log_{3-2\sqrt{2}}\left(27^{\log_9 2} + 2^{\log_8 27}\right)$

**7) G =** $\lg\left(25^{\log_5 6} + 49^{\log_7 8}\right) - e^{\ln 3}$ **8) H =** $9^{\frac{1}{\log_6 3}} + 4^{\frac{1}{\log_8 2}} - 10^{\log 99}$ **9) I =** $\lg\left(\sqrt{81^{\log_3 5} + 27^{\log_9 36}} + 3^{2\log_9 71}\right)$

**10) J =** $= 4^{1-2\log_2 \sqrt[4]{7}} - 36^{\log_6 2} + 81^{0,25-0,5\log_9 7}$ **11) K =** $\log_3(\log_2 8)$

**12) L =** $\log_{2013}\left\{\log_4(\log_2 256) - \log_{0,25}\left[\log_9(\log_4 64)\right]\right\}$ **13) M** $= \log_3 2.\log_4 3.\log_5 4.\log_6 5.\log_7 6.\log_8 7$

**14) N** $= \lg(\tan 1^0) + \lg(\tan 2^0) + ... + \lg(\tan 88^0) + \lg(\tan 89^0)$

**Giải:**

**1) A =** $\log_3\left(\log_{2\sqrt{2}} \sqrt[3]{\sqrt{2}}\right) = \log_3\left(\log_{2^{\frac{3}{2}}} 2^{\frac{1}{6}}\right) = \log_3\left(\dfrac{1}{6}.\dfrac{2}{3}\right) = \log_3 \dfrac{1}{9} = \log_3 3^{-2} = -2$

**2) B =** $\log_{\sqrt{6}} 3.\log_3 36 = \log_{\sqrt{6}} 36 = \log_{6^{\frac{1}{2}}} 6^2 = 4$

**3) C =** $\log_{\frac{1}{3}} 5.\log_{25} \dfrac{1}{27} = \log_{3^{-1}} 5.\log_{5^2} 3^{-3} = (-5).\left(-\dfrac{3}{2}\right).\log_3 5.\log_5 3 = \dfrac{15}{2}$

**4) D =** $\left(\sqrt[3]{9}\right)^{\frac{3}{2\log_5 3}} = \left(3^{\frac{2}{3}}\right)^{\frac{3\log_3 5}{2}} = 3^{\log_3 5} = 5$

**5) E** $= 25^{\frac{1}{2}+\frac{1}{9}\log_{\frac{1}{5}} 27+\log_{125} 81} = \left(5^2\right)^{\frac{1}{2}+\frac{1}{9}\log_{5^{-1}} 3^3+\log_{5^3} 3^4} = 5^{1-\frac{2}{3}\log_5 3+\frac{8}{3}\log_5 3} = 5^{1+2\log_5 3} = 5.5^{\log_5 3^2} = 5.9 = 45$

**6) F =** $\log_{3-2\sqrt{2}}\left(27^{\log_9 2}+2^{\log_8 27}\right)=\log_{3-2\sqrt{2}}\left[\left(3^3\right)^{\log_{3^2}2}+2^{\log_{2^3}3^3}\right]=\log_{3-2\sqrt{2}}\left(3^{\frac{3}{2}\log_3 2}+2^{\log_2 3}\right)$

$$=\log_{3-2\sqrt{2}}\left(3^{\log_3 2^{\frac{3}{2}}}+2^{\log_2 3}\right)=\log_{3-2\sqrt{2}}\left(2^{\frac{3}{2}}+3\right)=\log_{\left(3+2\sqrt{2}\right)^{-1}}\left(3+2\sqrt{2}\right)=-1$$

**7) G =** $\lg\left(25^{\log_5 6}+49^{\log_7 8}\right)-e^{\ln 3}=\lg\left[\left(5^2\right)^{\log_5 6}+\left(7^2\right)^{\log_7 8}\right]-3=\lg\left(5^{\log_5 6^2}+7^{\log_7 8^2}\right)-3$

$$=\lg\left(6^2+8^2\right)-3=\lg 10^2-3=2-3=-1$$

**8) H =** $9^{\frac{1}{\log_6 3}}+4^{\frac{1}{\log_8 2}}-10^{\log 99}=\left(3^2\right)^{\log_3 6}+\left(2^2\right)^{\log_2 8}-99=3^{\log_3 6^2}+2^{\log_2 8^2}-99=6^2+8^2-99=1$

**9) I =** $\lg\left(\sqrt{81^{\log_3 5}+27^{\log_9 36}}+3^{2\log_9 71}\right)=\lg\left(\sqrt{\left(3^4\right)^{\log_3 5}+\left(3^3\right)^{\log_{3^2}6^2}}+3^{2\log_{3^2}71}\right)$

$$=\lg\left(\sqrt{3^{\log_3 5^4}+3^{\log_3 6^3}}+3^{\log_3 71}\right)=\lg\left(\sqrt{5^4+6^3}+71\right)=\lg\left(29+71\right)=\lg 100=2$$

**10) J** $=4^{1-2\log_2\sqrt[4]{7}}-36^{\log_6 2}+81^{0,25-0,5\log_9 7}=\left(2^2\right)^{1-2\log_2\sqrt[4]{7}}+\left(6^2\right)^{\log_6 2}+\left(3^4\right)^{0,25-\frac{1}{2}.\log_{3^2}7}$

$$=\frac{2^2}{2^{4\log_2\sqrt[4]{7}}}+6^{\log_6 4}+\frac{3}{3^{\log_3 7}}=\frac{4}{7}-4+\frac{3}{7}=-3$$

**11) K =** $\log_3(\log_2 8)=\log_3\left(\log_2 2^3\right)=\log_3 3=1$

**12) L =** $\log_{2013}\left\{\log_4(\log_2 256)-\log_{0,25}\left[\log_9(\log_4 64)\right]\right\}=\log_{2013}\left\{\log_4(\log_2 2^8)-\log_{0,25}\left[\log_9(\log_4 4^3)\right]\right\}$

$$=\log_{2013}\left[\log_4 8-\log_{0,25}\left(\log_9 3\right)\right]=\log_{2013}\left(\log_{2^2}2^3-\log_{\left(\frac{1}{2}\right)^2}\frac{1}{2}\right)=\log_{2013}\left(\frac{3}{2}-\frac{1}{2}\right)=\log_{2013}1=0$$

**13) M** $=\log_3 2.\log_4 3.\log_5 4.\log_6 5.\log_7 6.\log_8 7=\log_8 7.\log_7 6.\log_6 5.\log_5 4.\log_4 3.\log_3 2=\log_8 2=\frac{1}{3}$

**14) N** $=\lg(\tan 1^0)+\lg(\tan 2^0)+...+\lg(\tan 88^0)+\lg(\tan 89^0)$

$$=\left[\lg(\tan 1^0)+\lg(\tan 89^0)\right]+\left[\lg(\tan 2^0)+\lg(\tan 88^0)\right]+...+\left[\lg(\tan 44^0)+\lg(\tan 46^0)\right]+\lg(\tan 45^0)$$

$$=\lg\left(\tan 1^0.\tan 89^0\right)+\lg\left(\tan 2^0.\tan 88^0\right)+...+\lg\left(\tan 44^0.\tan 46^0\right)+\lg\left(\tan 45^0\right)$$

$$=\lg\left(\tan 1^0.\cot 1^0\right)+\lg\left(\tan 2^0.\cot 2^0\right)+...+\lg\left(\tan 44^0.\cot 44^0\right)+\lg\left(\tan 45^0\right)$$

$$=\lg 1+\lg 1+...+\lg 1+\lg 1=0+0+...+0+0=0$$

**Ví dụ 2:** Đơn giản các biểu thức sau (giả sử các biểu thức đều có nghĩa):

**1) A =** $\log_a\left(a^2\sqrt[4]{a^3\sqrt[5]{a}}\right)$ **2) B =** $\left(\log_a b+\log_b a+2\right)\left(\log_a b-\log_{ab}b\right)\log_b a-1$

**3) C =** $\lg\left|\log_{\frac{1}{a^3}}\sqrt[5]{a\sqrt{a}}\right|$ **4) D =** $\dfrac{\log_2\left(2a^2\right)+\left(\log_2 a\right)a^{\log_a(\log_2 a+1)}+\frac{1}{2}\log_2^2 a^4}{\log_2 a^3.\left(3\log_2 a+1\right)+1}$

**Giải:**

**1) A =** $\log_a\left(a^2\sqrt[4]{a^3\sqrt[5]{a}}\right)=\log_a\left(a^2.\sqrt[4]{a^3.a^{\frac{1}{5}}}\right)=\log_a\left[a^2.\left(a^{\frac{16}{5}}\right)^{\frac{1}{4}}\right]=\log_a\left(a^2.a^{\frac{4}{5}}\right)=\log_a a^{\frac{14}{5}}=\frac{14}{5}$

**2) B**$=\left(\log_a b+\log_b a+2\right)\left(\log_a b-\log_{ab} b\right)\log_b a-1=\left(\log_a b+\frac{1}{\log_a b}+2\right)\left(\log_a b.\log_b a-\log_{ab} b.\log_b a\right)-1$

$$=\frac{\log_a^2 b+2\log_a b+1}{\log_a b}\left(1-\log_{ab} a\right)-1=\frac{\left(\log_a b+1\right)^2}{\log_a b}.\left(1-\frac{1}{\log_a ab}\right)-1$$

$$=\frac{\left(\log_a b+1\right)^2}{\log_a b}.\left(1-\frac{1}{1+\log_a b}\right)-1=\frac{\left(\log_a b+1\right)^2}{\log_a b}.\frac{\log_a b}{1+\log_a b}-1=\log_a b+1-1=\log_a b$$

**3) C =** $\lg\left|\log_{\frac{1}{a^3}}\sqrt[5]{a\sqrt{a}}\right|=\lg\left|\log_{\frac{1}{a^3}}\sqrt[5]{a.a^{\frac{1}{2}}}\right|=\lg\left|\log_{\frac{1}{a^3}}\left(a^{\frac{3}{2}}\right)^{\frac{1}{5}}\right|=\lg\left|\log_{a^{-3}} a^{\frac{3}{10}}\right|=\lg\left|-\frac{1}{10}\right|=\lg\frac{1}{10}=-1$

**4) D =** $\dfrac{\log_2\left(2a^2\right)+\left(\log_2 a\right)a^{\log_a\left(\log_2 a+1\right)}+\frac{1}{2}\log_2^2 a^4}{\log_2 a^3.\left(3\log_2 a+1\right)+1}=\dfrac{1+2\log_2 a+\log_2 a.\left(\log_2 a+1\right)+8\log_2^2 a}{3\log_2 a.\left(3\log_2 a+1\right)+1}$

$$=\frac{9\log_2^2 a+3\log_2 a+1}{9\log_2^2 a+3\log_2 a+1}=1$$

**Ví dụ 3:** Cho $\log_a b=3;\log_a c=-2$. Tính $\log_a x$ biết: **1)** $x=a^3b^2\sqrt{c}$ **2)** $x=\dfrac{a^4\sqrt[3]{b}}{c^3}$ **3)** $x=\log_a\dfrac{a^2\sqrt[3]{bc}}{\sqrt[3]{a\sqrt{c}b^3}}$

**Giải:** Cho $\log_a b=3$; $\log_a c=-2$

**1)** Với $x=a^3b^2\sqrt{c}$

$$\Rightarrow \log_a x=\log_a\left(a^3b^2\sqrt{c}\right)=\log_a a^3+\log_a b^2+\log_a c^{\frac{1}{2}}=3+2\log_a b+\frac{1}{2}\log_a c=3+2.3+\frac{1}{2}.(-2)=8$$

**2)** Với $x=\dfrac{a^4\sqrt[3]{b}}{c^3}$

$$\Rightarrow \log_a x=\log_a\frac{a^4\sqrt[3]{b}}{c^3}=\log_a a^4+\log_a b^{\frac{1}{3}}+\log_a c^3=4+\frac{1}{3}\log_a b+3\log_a c=4+\frac{1}{3}.3+3.(-2)=-1$$

**3)** Với $x=\log_a\dfrac{a^2\sqrt[3]{bc}}{\sqrt[3]{a\sqrt{c}b^3}}$

$$\Rightarrow \log_a x=\log_a\frac{a^2\sqrt[3]{bc}}{\sqrt[3]{a\sqrt{c}b^3}}=\log_a\frac{a^2b^{\frac{1}{3}}c}{a^{\frac{1}{3}}b^3c^{\frac{1}{6}}}=\log_a\frac{a^{\frac{5}{3}}c^{\frac{5}{6}}}{b^{\frac{8}{3}}}=\log_a a^{\frac{5}{3}}-\log_a b^{\frac{8}{3}}+\log_a c^{\frac{3}{2}}$$

$$=\frac{5}{3}-\frac{8}{3}\log_a b+\frac{5}{6}\log_a c=\frac{5}{3}-\frac{8}{3}.3+\frac{5}{6}(-2)=-8$$

**Ví dụ 4:** Hãy biểu diễn theo *a* ( hoặc cả *b* hoặc *c*) các biểu thức sau:

**1) A** = $\log_{20} 0{,}16$ biết $\log_2 5 = a$

**2) B** = $\log_{25} 15$ biết $\log_{15} 3 = a$

**3) C** = $\log 40$ biết $\log_{\sqrt{2}}\left(\dfrac{1}{\sqrt[3]{5}}\right) = a$

**4) D** = $\log_6(21{,}6)$ biết $\log_2 3 = a$ và $\log_2 5 = b$

**5) E** = $\log_{35} 28$ biết $\log_{14} 7 = a$ và $\log_{14} 5 = b$

**6) F** = $\log_{25} 24$ biết $\log_6 15 = a$ và $\log_{12} 18 = b$

**7) G** = $\log_{125} 30$ biết $\lg 3 = a$ và $\lg 2 = b$.

**8) H** = $\log_{\sqrt[3]{5}} \dfrac{49}{8}$ biết $\log_{25} 7 = a$ và $\log_2 5 = b$.

**9) I** = $\log_{140} 63$ biết $\log_2 3 = a$; $\log_3 5 = b$; $\log_2 7 = c$

**10) J** = $\log_6 35$ biết $\log_{27} 5 = a$; $\log_8 7 = b$; $\log_2 3 = c$

**Giải:**

**1) A =** $\log_{20} 0{,}16$ biết $\log_2 5 = a$. Ta có: **A =** $\log_{20} 0{,}04 = \log_{20}\dfrac{2}{5^3} = \dfrac{\log_2 \dfrac{2}{5^3}}{\log_2(2^2.5)} = \dfrac{1-3\log_2 5}{2+\log_2 5} = \dfrac{1-3a}{2+a}$

**2) B** = $\log_{25} 15$ biết $\log_{15} 3 = a$. Ta có: $a = \log_{15} 3 = \dfrac{1}{\log_3(3.5)} = \dfrac{1}{1+\log_3 5} \Rightarrow \log_3 5 = \dfrac{1}{a} - 1 = \dfrac{1-a}{a}$

$$\Rightarrow \mathbf{B} = \log_{25} 15 = \frac{\log_3 15}{\log_3 25} = \frac{\log_3(3.5)}{\log_3 5^2} = \frac{1+\log_3 5}{2\log_3 5} = \frac{1+\dfrac{1-a}{a}}{2.\dfrac{1-a}{a}} = \frac{1}{2(1-a)}$$

**3) C** = $\log 40$ biết $\log_{\sqrt{2}}\left(\dfrac{1}{\sqrt[3]{5}}\right) = a$. Ta có: $a = \log_{\sqrt{2}}\left(\dfrac{1}{\sqrt[3]{5}}\right) = \log_{2^{\frac{1}{2}}} 5^{-\frac{1}{3}} = -\dfrac{2}{3}\log_2 5 \Rightarrow \log_2 5 = -\dfrac{3a}{2}$

$$\Rightarrow \mathbf{C} = \log 40 = \frac{\log_2 40}{\log_2 10} = \frac{\log_2(2^3.5)}{\log_2(2.5)} = \frac{3+\log_2 5}{1+\log_2 5} = \frac{3-\dfrac{3a}{2}}{1-\dfrac{3a}{2}} = \frac{6-3a}{2-3a}$$

**4) D** = $\log_6(21{,}6)$ biết $\log_2 3 = a$ và $\log_2 5 = b$

Ta có: **D** = $\log_6(21{,}6) = \dfrac{\log_2(21{,}6)}{\log_2 6} = \dfrac{\log_2 \dfrac{2^2.3^3}{5}}{\log_2(2.3)} = \dfrac{2+3\log_2 3 - \log_2 5}{1+\log_2 3} = \dfrac{2+3a-b}{1+a}$

**5) E** = $\log_{35} 28$ biết $\log_{14} 7 = a$ và $\log_{14} 5 = b$

Ta có: $a = \log_{14} 7 = \dfrac{1}{\log_7(2.7)} = \dfrac{1}{1+\log_7 2} \Rightarrow \log_7 2 = \dfrac{1}{a} - 1 = \dfrac{1-a}{a}$

$b = \log_{14} 5 = \dfrac{\log_7 5}{\log_7(7.2)} = \dfrac{\log_7 5}{1+\log_7 2} \Rightarrow \log_7 5 = b(1+\log_7 2) = b.\left(1+\dfrac{1-a}{a}\right) = \dfrac{b}{a}$

$$\Rightarrow \mathbf{E} = \log_{35} 28 = \frac{\log_7 28}{\log_7 35} = \frac{\log_7(7.2^2)}{\log_7(7.5)} = \frac{1+2\log_7 2}{1+\log_7 5} = \frac{1+2.\dfrac{1-a}{a}}{1+\dfrac{b}{a}} = \frac{2-a}{a+b}$$

6) **F** = $\log_{25}24$ biết $\log_6 15 = a$ và $\log_{12}18 = b$

Ta có: $a = \log_6 15 = \dfrac{\log_2 15}{\log_2 6} = \dfrac{\log_2 3 + \log_2 5}{1+\log_2 3}$ (1) $\qquad b = \log_{12}18 = \dfrac{\log_2 18}{\log_2 12} = \dfrac{\log_2\left(2.3^2\right)}{\log_2\left(2^2.3\right)} = \dfrac{1+2\log_2 3}{2+\log_2 3}$ (2)

Từ (2) $\Rightarrow b(2+\log_2 3) = 1+2\log_2 3 \Leftrightarrow (b-2)\log_2 3 = 1-2b \Leftrightarrow \log_2 3 = \dfrac{1-2b}{b-2}$

Từ (1) $\Rightarrow \log_2 5 = a\left(1+\log_2 3\right) - \log_2 3 = \left(a-1\right)\log_2 3 + a = \left(a-1\right)\dfrac{1-2b}{b-2} + a = \dfrac{2b-a-ab-1}{b-2}$

$$\Rightarrow \mathbf{F} = \log_{25}24 = \frac{\log_2 24}{\log_2 25} = \frac{\log_2\left(2^3.3\right)}{\log_2 5^2} = \frac{3+\log_2 3}{2\log_2 5} = \frac{3+\dfrac{1-2b}{b-2}}{2.\dfrac{2b-a-ab-1}{b-2}} = \frac{b-5}{4b-2a-2ab-2}$$

7) **G** = $\log_{125}30$ biết $\lg 3 = a$ và $\lg 2 = b$.

Ta có: $b = \lg 2 = \lg\left(\dfrac{10}{5}\right) = 1-\lg 5 \Rightarrow \lg 5 = 1-b \Rightarrow \mathbf{G} = \log_{125}30 = \dfrac{\lg 30}{\lg 125} = \dfrac{\lg(3.10)}{\lg\left(5^3\right)} = \dfrac{1+\lg 3}{3\lg 5} = \dfrac{1+a}{3(1-b)}$

8) **H** = $\log_{\sqrt[3]{5}}\dfrac{49}{8}$ biết $\log_{25}7 = a$ và $\log_2 5 = b$.

Ta có: $a = \log_{25}7 = \dfrac{\log_2 7}{\log_2 25} = \dfrac{\log_2 7}{2\log_2 5} = \dfrac{\log_2 7}{2b} \Rightarrow \log_2 7 = 2ab$

$$\Rightarrow \mathbf{H} = \log_{\sqrt[3]{5}}\frac{49}{8} = \frac{\log_2\dfrac{49}{8}}{\log_2\sqrt[3]{5}} = \frac{\log_2\dfrac{7^2}{2^3}}{\log_2 5^{\frac{1}{3}}} = \frac{2\log_2 7-3}{\dfrac{1}{3}\log_2 5} = \frac{2.2ab-3}{\dfrac{1}{3}b} = \frac{12ab-9}{b}$$

9) **I** = $\log_{140}63$ biết $\log_2 3 = a$; $\log_3 5 = b$; $\log_2 7 = c$

Ta có : $\log_2 5 = \log_2 3.\log_3 5 = ab \Rightarrow \mathbf{I} = \log_{140}63 = \dfrac{\log_2 63}{\log_2 140} = \dfrac{\log_2\left(3^2.7\right)}{\log_2\left(2^2.5.7\right)} = \dfrac{2\log_2 3+\log_2 7}{2+\log_2 5+\log_2 7} = \dfrac{2a+c}{2+ab+c}$

10) **J** = $\log_6 35$ biết $\log_{27}5 = a$; $\log_8 7 = b$; $\log_2 3 = c$

$$\begin{cases} a = \log_{27}5 = \dfrac{\log_2 5}{\log_2 27} = \dfrac{\log_2 5}{3\log_2 3} = \dfrac{\log_2 5}{3c} \Rightarrow \log_2 5 = 3ac \\ b = \log_8 7 = \dfrac{\log_2 7}{\log_2 8} = \dfrac{\log_2 7}{3} \Rightarrow \log_2 7 = 3b \end{cases} \Rightarrow \mathbf{J} = \log_6 35 = \frac{\log_2 35}{\log_2 6} = \frac{\log_2 5+\log_2 7}{1+\log_2 3} = \frac{3ac+3b}{1+c}$$

**Ví dụ 5:** Tính giá trị của biểu thức:

**1) A =** $\log_{\frac{\sqrt{b}}{a}}\dfrac{\sqrt[3]{b}}{\sqrt{a}}$ biết $\log_a b = \sqrt{3}$.

**2) B =** $\dfrac{a^{\frac{1}{4}} - a^{\frac{9}{4}}}{a^{\frac{1}{4}} - a^{\frac{5}{4}}} - \dfrac{b^{-\frac{1}{2}} - b^{\frac{3}{2}}}{b^{\frac{1}{2}} + b^{-\frac{1}{2}}}$ biết $a = 2013-\sqrt{2}$; $b = \sqrt{2}-2012$

**Giải:**

**1)** **A =** $\log_{\frac{\sqrt{b}}{a}} \frac{\sqrt[3]{b}}{\sqrt{a}}$ biết $\log_a b = \sqrt{3}$.

$$A = \log_{\frac{\sqrt{b}}{a}} \frac{\sqrt[3]{b}}{\sqrt{a}} = \log_{\frac{\sqrt{b}}{a}} b^{\frac{1}{3}} - \log_{\frac{\sqrt{b}}{a}} a^{\frac{1}{2}} = \frac{1}{3\log_b \frac{\sqrt{b}}{a}} - \frac{1}{2\log_a \frac{\sqrt{b}}{a}} = \frac{1}{3\left(\frac{1}{2} - \log_b a\right)} - \frac{1}{2\left(\frac{1}{2}\log_a b - 1\right)}$$

$$= \frac{1}{3\left(\frac{1}{2} - \frac{1}{\log_a b}\right)} - \frac{1}{\log_a b - 2} = \frac{2\log_a b}{3(\log_a b - 2)} - \frac{1}{\log_a b - 2} = \frac{2\log_a b - 3}{3(\log_a b - 2)} = \frac{2\sqrt{3} - 3}{3(\sqrt{3} - 2)} = -\frac{\sqrt{3}}{3}$$

**2)** **B =** $\frac{a^{\frac{1}{4}} - a^{\frac{9}{4}}}{a^{\frac{1}{4}} - a^{\frac{5}{4}}} - \frac{b^{-\frac{1}{2}} - b^{\frac{3}{2}}}{b^{\frac{1}{2}} + b^{-\frac{1}{2}}}$ biết $a = 2013 - \sqrt{2}$; $b = \sqrt{2} - 2012$

$$B = \frac{a^{\frac{1}{4}} - a^{\frac{9}{4}}}{a^{\frac{1}{4}} - a^{\frac{5}{4}}} - \frac{b^{-\frac{1}{2}} - b^{\frac{3}{2}}}{b^{\frac{1}{2}} + b^{-\frac{1}{2}}} = \frac{a^{\frac{1}{4}}(1 - a^2)}{a^{\frac{1}{4}}(1 - a)} - \frac{b^{-\frac{1}{2}}(1 - b^2)}{b^{-\frac{1}{2}}(1 + b)} = (1 + a) - (1 - b) = a + b = 2013 - \sqrt{2} + \sqrt{2} - 2012 = 1$$

**Ví dụ 6**: Chứng minh rằng (với giả thiết các biểu thức đều có nghĩa):

**1)** $\log_{ac}(bc) = \frac{\log_a b + \log_a c}{1 + \log_a c}$ **2)** $a^{\log_b c} = c^{\log_b a}$ **3)** Nếu $4a^2 + 9b^2 = 4ab$ thì $\lg\frac{2a + 3b}{4} = \frac{\lg a + \lg b}{2}$

**4)** Nếu $a^2 + 4b^2 = 12ab$ thì $\log_{2013}(a + 2b) - 2\log_{2013} 2 = \frac{1}{2}(\log_{2013} a + \log_{2013} b)$

**5)** Nếu $a = 10^{\frac{1}{1 - \lg b}}$; $b = 10^{\frac{1}{1 - \lg c}}$ thì $c = 10^{\frac{1}{1 - \lg a}}$ **6)** Nếu $a = \log_{12} 18$; $b = \log_{24} 54$ thì: $ab + 5(a - b) = 1$

**7)** $\log_a^2 \frac{b}{c} = \log_a^2 \frac{c}{b}$ **8)** Trong 3 số: $\log_{\frac{a}{b}}^2 \frac{c}{b}; \log_{\frac{b}{c}}^2 \frac{a}{c}$ và $\log_{\frac{c}{a}}^2 \frac{b}{a}$ luôn có ít nhất một số lớn hơn 1.

**Giải:**

**1)** $\log_{ac}(bc) = \frac{\log_a b + \log_a c}{1 + \log_a c}$. Ta có: $\frac{\log_a b + \log_a c}{1 + \log_a c} = \frac{\log_a bc}{\log_a a + \log_a c} = \frac{\log_a(bc)}{\log_a(ac)} = \log_{ac}(bc)$ (đpcm)

**2)** $a^{\log_b c} = c^{\log_b a}$. Đặt $a^{\log_b c} = t \Rightarrow \begin{cases} a^{\log_b c} = a^t \\ c = b^t \to c^{\log_b a} = b^{t\log_b a} = b^{\log_b a^t} = a^t \end{cases} \Rightarrow a^{\log_b c} = c^{\log_b a}$ (đpcm)

**3)** Nếu $4a^2 + 9b^2 = 4ab$ thì $\lg\frac{2a + 3b}{4} = \frac{\lg a + \lg b}{2}$

Ta có: $4a^2 + 9b^2 = 4ab \Leftrightarrow 4a^2 + 12ab + 9b^2 = 16ab \Leftrightarrow (2a + 3b)^2 = 16ab \Leftrightarrow \left(\frac{2a + 3b}{4}\right)^2 = ab$

$\Rightarrow \lg\left(\frac{2a + 3b}{4}\right)^2 = \lg(ab) \Leftrightarrow 2\lg\frac{2a + 3b}{4} = \lg a + \lg b \Leftrightarrow \lg\frac{2a + 3b}{4} = \frac{\lg a + \lg b}{2}$ (đpcm)

**4)** Nếu $a^2+4b^2=12ab$ thì $\log_{2013}(a+2b)-2\log_{2013}2=\frac{1}{2}(\log_{2013}a+\log_{2013}b)$

Ta có: $a^2+4b^2=12ab \Leftrightarrow a^2+4ab+4b^2=16ab \Leftrightarrow (a+2b)^2=16ab \Leftrightarrow \left(\frac{a+2b}{4}\right)^2=ab$

$$\Rightarrow \log_{2013}\left(\frac{a+2b}{4}\right)^2=\log_{2013}(ab) \Leftrightarrow 2\left[\log_{2013}(a+2b)-2\log_{2013}2\right]=\log_{2013}a+\log_{2013}b$$

$$\Leftrightarrow \log_{2013}(a+2b)-2\log_{2013}2=\frac{1}{2}(\log_{2013}a+\log_{2013}b) \quad \text{(đpcm)}$$

**5)** Nếu $a=10^{\frac{1}{1-\lg b}}$; $b=10^{\frac{1}{1-\lg c}}$ thì $c=10^{\frac{1}{1-\lg a}}$

Ta có: $a=10^{\frac{1}{1-\lg b}} \Leftrightarrow \lg a=\lg 10^{\frac{1}{1-\lg b}}=\frac{1}{1-\lg b} \Leftrightarrow \lg b=1-\frac{1}{\lg a}=\frac{\lg a-1}{\lg a} \quad (1)$

$b=10^{\frac{1}{1-\lg c}} \Leftrightarrow \lg b=\lg 10^{\frac{1}{1-\lg c}}=\frac{1}{1-\lg c} \quad (2)$

Từ (1) và (2) $\Rightarrow \frac{\lg a-1}{\lg a}=\frac{1}{1-\lg c} \Leftrightarrow \lg c=1-\frac{\lg a}{\lg a-1}=\frac{1}{1-\lg a} \Rightarrow 10^{\lg c}=10^{\frac{1}{1-\lg a}} \Leftrightarrow c=10^{\frac{1}{1-\lg a}}$ (đpcm).

**6)** Nếu $a=\log_{12}18$; $b=\log_{24}54$ thì: $ab+5(a-b)=1$

Ta có: $a=\log_{12}18=\frac{\log_2 18}{\log_2 12}=\frac{\log_2(2.3^2)}{\log_2(2^2.3)}=\frac{1+2\log_2 3}{2+\log_2 3} \Rightarrow a(2+\log_2 3)=1+2\log_2 3 \Leftrightarrow \log_2 3=\frac{1-2a}{a-2} \quad (1)$

$b=\log_{24}54=\frac{\log_2 54}{\log_2 24}=\frac{\log_2(2.3^3)}{\log_2(2^3.3)}=\frac{1+3\log_2 3}{3+\log_2 3} \Rightarrow b(3+\log_2 3)=1+3\log_2 3 \Leftrightarrow \log_2 3=\frac{1-3b}{b-3} \quad (2)$

Từ (1) và (2) $\Rightarrow \frac{1-2a}{a-2}=\frac{1-3b}{b-3} \Leftrightarrow (1-2a)(b-3)=(1-3b)(a-2) \Leftrightarrow ab+5(a-b)=1$ (đpcm)

**7)** $\log_a^2\frac{b}{c}=\log_a^2\frac{c}{b}$

Ta có : $\log_a^2\frac{b}{c}=\left(\log_a\frac{b}{c}\right)^2=\left[\log_a\left(\frac{c}{b}\right)^{-1}\right]^2=\left(-\log_a\frac{c}{b}\right)^2=\left(\log_a\frac{c}{b}\right)^2=\log_a^2\frac{c}{b}$ (đpcm)

**8)** Trong ba số: $\log_{\frac{a}{b}}^2\frac{c}{b}$; $\log_{\frac{b}{c}}^2\frac{a}{c}$ và $\log_{\frac{c}{a}}^2\frac{b}{a}$ luôn có ít nhất một số lớn hơn 1.

Áp dụng công thức ở ý **7)** ta có: $\log_{\frac{a}{b}}^2\frac{c}{b}=\log_{\frac{a}{b}}^2\frac{b}{c}$ ; $\log_{\frac{b}{c}}^2\frac{a}{c}=\log_{\frac{b}{c}}^2\frac{c}{a}$ ; $\log_{\frac{c}{a}}^2\frac{b}{a}=\log_{\frac{c}{a}}^2\frac{a}{b}$

$$\Rightarrow \log_{\frac{a}{b}}^2\frac{c}{b}.\log_{\frac{b}{c}}^2\frac{a}{c}.\log_{\frac{c}{a}}^2\frac{b}{a}=\log_{\frac{a}{b}}^2\frac{b}{c}.\log_{\frac{b}{c}}^2\frac{c}{a}.\log_{\frac{c}{a}}^2\frac{a}{b}=\left(\log_{\frac{a}{b}}\frac{b}{c}.\log_{\frac{b}{c}}\frac{c}{a}.\log_{\frac{c}{a}}\frac{a}{b}\right)^2=1^2=1$$

$\Rightarrow$ Trong ba số không âm: $\log_{\frac{a}{b}}^2\frac{c}{b}$; $\log_{\frac{b}{c}}^2\frac{a}{c}$ và $\log_{\frac{c}{a}}^2\frac{b}{a}$ luôn có ít nhất một số lớn hơn 1.

## B. BÀI LUYỆN

**Bài 1**: Tính giá trị các biểu thức sau:

1) **A =** $\log_{\frac{1}{25}} 5\sqrt[4]{5}$

2) **B =** $\log_2 8.\log_{\frac{1}{8}} 4$

3) **C =** $\log_{\sqrt{3}} \frac{1}{9}.\log_{\frac{1}{5}}\left(5\sqrt{5}\right)$

4) **D =** $5^{3-2\log_5 4}$

5) **E =** $9^{\frac{1}{2}\log_3 2-2\log_{27} 3}$

6) **F =** $4^{\log_2 3}+9^{\log_{\sqrt{3}} 2}$

7) **G =** $\dfrac{25^{\log_5 6}+49^{\log_7 8}-3}{3^{1+\log_9 4}+4^{2-\log_2 3}-5^{\log_{125} 27}}$

8) **H =** $\log_3 6.\log_8 9.\log_6 2$

9) **I** $=\dfrac{\log_3 4.\log_6 8}{\log_6 4.\log_9 8}$

10) **J** $=2\log_{\frac{1}{3}} 6-\frac{1}{2}\log_{\frac{1}{3}} 400+3\log_{\frac{1}{3}} \sqrt[3]{45}$

11) **J** $=\dfrac{(27^{\frac{1}{\log_2 3}}+5^{\log_{25} 49})(81^{\frac{1}{\log_4 9}}-8^{\log_4 9})}{3+5^{\frac{1}{\log_{16} 25}}.5^{\log_5 3}}$

12) **K** $=\log_6 \frac{1}{3}+\log_6 \frac{1}{12}-27^{\log_3 5}-\sqrt{\log_{\frac{1}{2}}^2 16+9^{\frac{1}{\log_7 3}}+4^{\frac{1}{\log_9 2}}+\log_3 \tan\frac{\pi}{4}}$

**Bài 2:** Đơn giản các biểu thức sau (giả sử các biểu thức đều có nghĩa):

1) **A =** $\sqrt{\log_a b+\log_b a+2}\left(\log_a b-\log_{ab} b\right)\log_b a$

2) **B =** $\dfrac{\log_{a^3} a.\log_{\sqrt[3]{a}} a^4}{\log_{\frac{1}{a}} a^2}$

**Bài 3:** Hãy biểu diễn theo *a* ( hoặc cả *b* hoặc *c*) các biểu thức sau:

1) **A =** $\log_{\frac{1}{2}} 28$ biết $\log_7 2=a$

2) **B =** $\log_6 16$ biết $\log_{12} 27=a$.

3) **C =** $\log_{49} 32$ biết $\log_2 14=a$

4) **D =** $\log_{54} 168$ biết $\log_7 12=a$ và $\log_{12} 24=b$

5) **E =** $\log_{30} 1350$ biết $\log_{30} 3=a$ và $\log_{30} 5=b$

6) **F =** $\log_{\sqrt[3]{7}} \frac{121}{8}$ biết $\log_{49} 11=a$ và $\log_2 7=b$.

7) **G =** $\log_3 135$ biết $\log_2 5=a$ và $\log_2 3=b$.

**Bài 4:** Tính giá trị của biểu thức:

1) **A =** $\log_{\sqrt{ab}} \frac{b}{\sqrt{a}}$ biết $\log_a b=\sqrt{5}$.

2) **B =** $c^{\log_{\sqrt{c}}\left(\log_{\sqrt{a}}\left(a\sqrt{b}\sqrt[3]{c}\right)\right)}$ biết $\log_a b=5$ và $\log_a c=3$

**Bài 5**: Chứng minh rằng (với giả thiết các biểu thức đều có nghĩa):

1) $\dfrac{\log_a c}{\log_{ab} c}=1+\log_a b$

2) Nếu $a^2+b^2=c^2$ thì $\log_{b+c} a+\log_{c-b} a=2\log_{c+b} a.\log_{c-b} a$

3) Nếu $a^2+b^2=7ab$ thì $\log_7 \frac{a+b}{3}=\frac{1}{2}(\log_7 a+\log_7 b)$

4) Nếu $a^2+9b^2=10ab$ thì $\log(a-3b)-\log 2=\frac{1}{2}(\log a+\log b)$